Mathematical Sciences and Applications
ISSN(Print): 2958-2903 | ISSN(Online): 2959-3220
Volume 2, Issue 2, 2023

Unified Framework of Stationary and Non-Stationary Subdivision Schemes for Curve Modeling

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History

Article ID	MSA-2023-15
Received	17.07.2023
Accepted	13.12.2023

Keywords:
Binary schemes, stationary and non-stationary schemes; continuity, generation and reproduction; limit stencil, approximating shapes.

Abstract: In this study, we explore the interaction of two Laurent polynomials to generate a new polynomial through multiplication. Introducing a parameter allows us to adjust the number of factors within this polynomial, thereby facilitating the creation of a diverse family of non-stationary subdivision schemes. Each distinct value of the parameter corresponds to a unique member of this scheme family. Additionally, leveraging the concept of asymptotic equivalence, we derive a separate family of stationary schemes. Following the derivation of stationary schemes, we proceed to analyze their key characteristics and practical applications. This includes detailed examination of properties such as convergence behavior, polynomial generation degree, reproduction capabilities, continuity, and the structure of limit stencils. Furthermore, we explore the practical applications of these schemes in generating smooth curves.

1 Introduction

Subdivision schemes are advanced techniques succeeding earlier subdivision methods, playing a pivotal role in Computer-Aided Geometric Design (CAGD). They offer efficient and powerful methods for creating curves and 3D structures from an initial control polygon. These schemes iteratively insert new control points between existing ones to refine polygons, ultimately producing smooth curves. Subdivision schemes are broadly categorized into interpolating schemes, which retain points from previous iterations, and approximating schemes, which do not. Approximating schemes are generally preferred for their computational efficiency. Subdivision schemes can also be classified as stationary, where refinement rules remain consistent across all levels, and non-stationary, where rules vary between levels. The concept of subdivision schemes was

introduced in 1956 by Georges de Rham, who developed methods ensuring C^0-continuity [24]. Subsequent advancements by researchers such as Chaikin [7], Catmull and Clark [8], and Doo and Sabin [9] refined these methods, leading to the construction of uniform B-splines and other smooth surfaces.
In the late 20th and early 21st centuries, further research expanded the applications and theoretical foundations of subdivision schemes. Scholars including Dyn and Levin [13] explored non-stationary schemes such as exponential splines and up-functions, demonstrating their equivalence and convergence properties compared to stationary schemes. These innovations contributed to the development of schemes capable of producing C^2-continuous curves and surfaces, often incorporating tension parameters to effectively control shape variations [6, 12]. Beccari et al. [4, 5], and Daniel et al. [10, 11] presented non-stationary subdivision schemes with parameters. Li et al. [15], Jena et al. [16] and Levin [17] also introduced non-stationary schemes. Conti et al. [20, 21] and Ghaffar et al. [22] also introduced the families of the non-stationary schemes.Saddiqi et al. [25, 26, 27], Some non-stationary subdivision schemes are also presented in the references [2, 23, 28]. Bari et al. [3] presented the construction and analysis of unified 4-point interpolating non-stationary subdivision surfaces in 2021. Mustafa [18] introduced the statistical and geometrical way of model selection for a family of subdivision schemes.
This paper introduces a unified family encompassing both stationary and non-stationary schemes. Stationary schemes are traditionally employed for precise curve modeling applications. In contrast, non-stationary schemes are tailored for generating fundamental shapes such as conic sections.
The structure of the paper is as follows: In Section 2, we outline the framework for deriving a family of non-stationary binary subdivision schemes. Subsequently, we induce a family of stationary binary subdivision schemes from the non-stationary family. The analysis of a specific member of the scheme family is discussed in Section 3. Section 4 is devoted to the application of the schemes, while the conclusions are presented in Section 5.

2 A framework for the generation of non-stationary and stationary schemes

In this section, we outline the introduction of non-stationary schemes derived from polynomials. Using the asymptotic equivalence relation, these non-stationary schemes are then transformed into stationary schemes. A specific non-stationary and stationary scheme has been derived. Similarly, other members of non-stationary and stationary schemes can also be derived.
Consider the following Laurent polynomials.

$$a(z) = \left(\frac{1}{9} - \frac{1}{2}\eta^k\right) + \left(\frac{7}{9} + \eta^k\right) z + \left(\frac{1}{9} - \frac{1}{2}\eta^k\right) z^2 \quad (1)$$

$$b(z) = \frac{(1+z)^{x+1}}{2^{x-1}(2z^2)}, \qquad x \geq 1 \quad (2)$$

after multiplying (1) and (2), we get

$$\alpha_x(z) \;=\; \frac{1}{2}\frac{\left(\left(\frac{1}{9}-\frac{1}{2}\eta^k\right)+\left(\frac{7}{9}+\eta^k\right)z+\left(\frac{1}{9}-\frac{1}{2}\eta^k\right)z^2\right)(1+z)^{x+1}}{2^{x-1}(2z^2)} \tag{3}$$

where

$$\eta^k = \frac{2\eta}{v^{k+1}(v^{k+1}+1)} \tag{4}$$

with

$$v^k = \frac{1}{2}\left(e^{i(\frac{t}{2^{k+1}})}+e^{-i(\frac{t}{2^{k+1}})}\right) = \cos\left(\frac{t}{2^{k+1}}\right), \qquad t \in [0,\Pi]\cup iR^+ \tag{5}$$

and

$$v^0 = \cos(\frac{t}{2}) = \begin{cases} \cos(\frac{\beta}{2}) \in (0,1) & if \quad t=\beta, \beta \in (0,pi) \\ 1 & if \quad t=0 \\ \cosh(\frac{\beta}{2}) \in (1,+\infty) & if \quad t=i\beta, \beta\in R^+ \end{cases}$$

we also define

$$v^{k+1} = \left(\frac{1+v^k}{2}\right)^{\frac{1}{2}}.$$

Remember that, if $t=0$ in (5) then

$$v^k = v^{k+1} = 1.$$

Similarly, we have

$$\lim_{k\to+\infty} v^k = \lim_{k\to+\infty} v^{k+1} = 1. \tag{6}$$

This is called asymptotically equivalent relation between two levels of iterations. From polynomial (3), a family of non-stationary subdivision schemes can be derived using different values of the parameter x. This family can then be transformed into a family of stationary schemes using the equivalence relation.

2.1 Derivation of specific non-stationary and stationary schemes

As an example, a specific non-stationary scheme is first derived at $x=1$. This scheme is then transformed into a stationary scheme using the asymptotic equivalence relation. Now, by substituting the value $x=1$ into (3), we get

$$\alpha_1(z) = \left(\frac{1}{18}-\frac{1}{4}\eta^k\right)z^2+\frac{1}{2}z+\frac{8}{9}+\frac{1}{2}\eta^k+\frac{1}{2}z^{-1}+\left(\frac{1}{18}-\frac{1}{4}\eta^k\right)z^{-2}. \tag{7}$$

By using the coefficients from (7), we can express two refinement rules in the following form:

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{2}f_{i-1}^{k}+\frac{1}{2}f_{i}^{k}. \\ f_{2i+1}^{k+1} &= \Big(\frac{1}{18}-\frac{1}{4}\eta^{k}\Big)f_{i-1}^{k}+\Big(\frac{8}{9}+\frac{1}{2}\eta^{k}\Big)f_{i}^{k}+\Big(\frac{1}{18}-\frac{1}{4}\eta^{k}\Big)f_{i+1}^{k}. \end{aligned}$$

The point f_{2i}^{k+1} is obtained using an affine combination of two points at the k-th level, while the point f_{2i+1}^{k+1} is obtained using an affine combination of three points from the k-th level.
After putting the value of η^{k}, we have

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{2}f_{i-1}^{k}+\frac{1}{2}f_{i}^{k}. \\ f_{2i+1}^{k+1} &= \Big(\frac{1}{18}-\frac{1}{4}\frac{2\eta}{\upsilon^{k+1}(\upsilon^{k+1}+1)}\Big)f_{i-1}^{k}+\Big(\frac{8}{9}+\frac{1}{2}\frac{2\eta}{\upsilon^{k+1}(\upsilon^{k+1}+1)}\Big)f_{i}^{k} \\ &\quad+\Big(\frac{1}{18}-\frac{1}{4}\frac{2\eta}{\upsilon^{k+1}(\upsilon^{k+1}+1)}\Big)f_{i+1}^{k}. \end{aligned}$$

By simplifying, we get

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{2}f_{i-1}^{k}+\frac{1}{2}f_{i}^{k}. \\ f_{2i+1}^{k+1} &= \frac{1}{2\upsilon^{k+1}(\upsilon^{k+1}+1)}\Big[\Big(\frac{1}{9}\upsilon^{k+1}(\upsilon^{k+1}+1)-\eta\Big)f_{i-1}^{k}+\Big(\frac{16}{9}\upsilon^{k+1}(\upsilon^{k+1}+1)+\Big)f_{i}^{k}+ \\ &\quad \Big(\frac{1}{9}\upsilon^{k+1}(\upsilon^{k+1}+1)-\eta\Big)f_{i+1}^{k}\Big]. \end{aligned} \tag{8}$$

Now we convert the non-stationary binary subdivision scheme (8) into a stationary subdivision scheme. For this, we use the asymptotically equivalent relation (6).

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{4}\Big[2f_{i-1}^{k}+2f_{i}^{k}\Big]. \\ f_{2i+1}^{k+1} &= \frac{1}{4}\Big[\Big(\frac{2}{9}-\eta\Big)f_{i-1}^{k}+\Big(\frac{32}{9}+\eta\Big)f_{i}^{k}+\Big(\frac{2}{9}-\eta\Big)f_{i+1}^{k}\Big]. \end{aligned} \tag{9}$$

The scheme is known as the 3-point binary parametric stationary subdivision scheme, which introduces new points based on up to three existing points. The scheme is called binary because it has two rules. The coefficients in this scheme, written in the following form, are called the mask of the scheme.

$$\alpha_1=\frac{1}{4}\Big[\frac{2}{9}-\eta,2,\frac{32}{9}+\eta,2,\frac{2}{9}-\eta\Big]. \tag{10}$$

The following polynomial is called a Laurent polynomial, where the variable can have integer powers that are positive, negative, or zero

$$\alpha_1(z)=\frac{(1+z)^2}{18z^2}\Big[1-\frac{9}{2}\eta+\Big(7+9\eta\Big)z+\Big(1-\frac{9}{2}\eta\Big)z^2\Big]. \tag{11}$$

The same approach can be applied to derive an infinite variety of non-stationary and stationary schemes at different integer values of the parameter x.

3 Analysis of the 3-point stationary scheme

In this section, we will discuss some important applications of the binary subdivision scheme, such as convergence, generation of polynomial degree, polynomial reproduction, continuity, and limit stencil.

3.1 Convergence of the 3-point stationary scheme

The scheme is convergent if its Laurent polynomial satisfies the conditions $\alpha(-1) = 0$, $\alpha(1) = 2$, which implies $\alpha(z) = (1+z)b(z)$, $b(1) = 1$. Therefore, we have the following result. The 3-point binary stationary subdivision scheme (9) satisfies the necessary and sufficient conditions of convergence. After substituting z=1,-1 in (11), we get $\alpha_1(1) = 2$ and $\alpha_1(-1) = 0$. Now we can rewrite polynomial (11), as

$$\alpha_1(z) = (1+z)b(z),$$

where

$$b(z) = \frac{(1+z)}{18z^2}\Big[1 - \frac{9}{2}\eta + \Big(7 + 9\eta\Big)z + \Big(1 - \frac{9}{2}\eta\Big)z^2\Big]. \tag{12}$$

Now, substituting $z = 1$ into (12), we obtain $b(1) = 1$. Thus, the scheme satisfies the necessary and sufficient conditions for convergence.

3.2 Degree of polynomial generation and reproduction

Here, we will discuss the subdivision scheme's ability for polynomial generation and reproduction. The methodology introduced by [19] is used to discuss the polynomial generation and reproduction property of the scheme. The degree of polynomial generation of the 3-point binary stationary subdivision scheme (9) is 1. The Laurent polynomial of the scheme is

$$\alpha_1(z) = \frac{(1+z)^2}{18z^2}\Big[1 - \frac{9}{2}\eta + \Big(7 + 9\eta\Big)z + \Big(1 - \frac{9}{2}\eta\Big)z^2\Big].$$

By factoring out the maximum common factor $1+z$, we obtain

$$\alpha_1(z) = (1+z)^{d+1}b(z), \quad d = 1,$$

where

$$b(z) = \frac{1}{18z^2}\Big[1 - \frac{9}{2}\eta + \Big(7 + 9\eta\Big)z + \Big(1 - \frac{9}{2}\eta\Big)z^2\Big].$$

Therefore, the degree of polynomial generation is d=1. The 3-point binary stationary subdivision scheme reproduces the polynomial of degree $d=1$ with respect to the parametrization $\tau=\frac{\alpha_1'(1)}{2}$ if and only if

$$\text{Case-I:} \quad \alpha_1^k(-1) \;=\; 0,, \quad k=0,1. \tag{13}$$

$$\text{Case-II:} \quad \alpha_1^k(1) \;=\; 2\prod_{l=0}^{k-1}(\tau-l), \quad k=0,1. \tag{14}$$

Consider

$$\alpha_1(z) \;=\; \frac{(1+z)^2}{18z^2}\Big[1-\frac{9}{2}\eta+\Big(7+9\eta\Big)z+\Big(1-\frac{9}{2}\eta\Big)z^2\Big].$$

Taking derivative, we get

$$\begin{aligned}\alpha_1'(z) \;=\; & \frac{1}{18}\frac{(1+z)\Big[2-9\eta+(14+18\eta)z+(2-9\eta)z^2\Big]}{z^2} \\ & +\frac{1}{36}\frac{(1+z)^2\Big[14+18\eta+2(2-9\eta)z\Big]}{z^2} \\ & -\frac{1}{18}\frac{(1+z)^2\Big[2-9\eta+(14+18\eta)z+(2-9\eta)z^2\Big]}{z^3}.\end{aligned} \tag{15}$$

Substituting $z=1$ into (15), we obtain $\alpha_1'(1)=0$. This implies

$$\tau \;=\; \frac{\alpha_1'(1)}{2}=\frac{0}{2}=0.$$

For $k=0,1$, it is easy to see that Case-I, i.e., (13), holds. Similarly, by manipulating simple algebraic operations, it is easy to see that Case-II, i.e., (14), is also satisfied. This completes the proof.

3.3 Continuity analysis of the 3-point stationary scheme

By following the methods of Dyn [14], the continuity analysis of the 3-point stationary scheme is discussed. The 3-point binary stationary subdivision scheme (9) is C^1-continuous. The Laurent polynomial of the scheme defined in (11) can be written as

$$\begin{aligned}\alpha_{1,1}(z) \;&=\; \Big(\frac{2z}{1+z}\Big)\alpha_1(z), \\ \alpha_{1,1}(z) \;&=\; \Big(\frac{2z}{1+z}\Big)\frac{(1+z)^2}{18z^2}\Big[1-\frac{9}{2}\eta+\Big(7+9\eta\Big)z+\Big(1-\frac{9}{2}\eta\Big)z^2\Big].\end{aligned}$$

After simplifying

$$\alpha_{1,1}(z) \;=\; \frac{1+z}{9z}\Big[1-\frac{9}{2}\eta+\Big(7+9\eta\Big)z+\Big(1-\frac{9}{2}\eta\Big)z^2\Big].$$

By collecting the terms of z, we have

$$\alpha_{1,1}(z) = \Big(\frac{1}{9}-\frac{1}{2}\eta\Big)z^2+\Big(\frac{8}{9}+\frac{1}{2}\eta\Big)z+\frac{8}{9}+\frac{1}{2}\eta+\Big(\frac{1}{9}-\frac{1}{2}\eta\Big)z^{-1},$$

and let the mask of the scheme S_1 corresponding to the polynomial $\alpha_{1,1}(z)$ be

$$\alpha_{1,1} = \Big[\frac{1}{9}-\frac{1}{2}\eta,\frac{8}{9}+\frac{1}{2}\eta,\frac{8}{9}+\frac{1}{2}\eta,\frac{1}{9}-\frac{1}{2}\eta\Big]. \tag{16}$$

The scheme (9) corresponding to $\alpha_1(z)$ is C^0 continuous if $\left\|\frac{1}{2}S_1\right\|_\infty < 1$, for this we have to check that

$$\left\|\frac{1}{2}S_1\right\|_\infty = \max\Big\{\frac{1}{2}\sum_j\left|\alpha_{1,1}^{2j}\right|,\quad \frac{1}{2}\sum_j\left|\alpha_{1,1}^{2j+1}\right|\Big\} < 1.$$

From (16), we get

$$\left\|\frac{1}{2}S_1\right\|_\infty = \max\Big\{\left|\frac{8}{18}+\frac{1}{4}\eta\right|+\left|\frac{1}{18}-\frac{1}{4}\eta\right|,\left|\frac{1}{18}-\frac{1}{4}\eta\right|+\left|\frac{8}{18}+\frac{1}{4}\eta\right|\Big\}.$$

After simplifying,

$$\left\|\frac{1}{2}S_1\right\|_\infty = \max\Big\{\frac{1}{2},\frac{1}{2}\Big\}=\frac{1}{2}<1.$$

The scheme S_1 is contractive, this implies that the scheme S i.e., (9) has C^0 continuity.
For C^1 continuity, the Laurent polynomial of the scheme S_2 can be written as

$$\alpha_{1,2}(z) = \Big(\frac{2z}{1+z}\Big)\alpha_1^1(z),$$
$$\alpha_{1,2}(z) = \Big(\frac{2z}{1+z}\Big)\frac{1+z}{9z}\Big[1-\frac{9}{2}\eta+\Big(7+9\eta\Big)z+\Big(1-\frac{9}{2}\eta\Big)z^2\Big].$$

and let the mask of the scheme S_2 corresponding to the polynomial $\alpha_{1,2}(z)$ be

$$\alpha_{1,2} = \Big[\frac{2}{9}-\eta,\frac{14}{9}+2\eta,\frac{2}{9}-\eta\Big]. \tag{17}$$

The scheme S_2 corresponding to $\alpha_{1,2}(z)$ is C^1 continuity if $\left\|\frac{1}{2}S_2\right\|_\infty < 1$, for this, we have to check

$$\left\|\frac{1}{2}S_2\right\|_\infty = \max\Big\{\frac{1}{2}\sum_j\left|\alpha_{1,2}^{2j}\right|,\quad \frac{1}{2}\sum_j\left|\alpha_{1,2}^{2j+1}\right|\Big\} < 1.$$

From (17)

$$
\begin{aligned}
\left\|\frac{1}{2}S_2\right\|_\infty &= \max\left\{\left|\frac{7}{9}+\frac{1}{2}\eta\right|, \left|\frac{1}{9}-\frac{1}{2}\eta\right| + \left|\frac{1}{9}-\frac{1}{2}\eta\right|\right\},\\
&= \max\left\{\left|\frac{7}{9}+\frac{1}{2}\eta\right|, \left|-\frac{1}{9}+\frac{1}{2}\eta\right| + \left|-\frac{1}{9}+\frac{1}{2}\eta\right|\right\},\\
&= \max\left\{\left|\frac{7}{9}+\eta\right|, \left|-\frac{2}{9}+\eta\right|\right\}.
\end{aligned}
$$

This implies that $\left\|\frac{1}{2}S_2\right\|_\infty < 1$ for $\eta \in \left(-\frac{7}{9}, \frac{2}{9}\right)$. Hence, the scheme S_2 is contractive, which implies that the scheme is at least C^1 continuous for $\eta \in \left(-\frac{7}{9}, \frac{2}{9}\right)$. This is what is required.

3.4 Limit stencil of the 3-point stationary scheme

In this section, we present the limit position of the control points of the initial control polygon at the limit curve using the limit stencil. For this purpose put $i = -1, 0,$ in (9) and get the following

$$
\begin{aligned}
f_{-2}^{k+1} &= \frac{1}{4}\Big[2f_{-1}^k + 2f_{-1}^k\Big]\\
f_{-1}^{k+1} &= \frac{1}{4}\Big[\Big(\frac{2}{9}-\eta\Big)f_{-2}^k + \Big(\frac{32}{9}+\eta\Big)f_{-1}^k + \Big(\frac{2}{9}-\eta\Big)f_0^k\Big]
\end{aligned}
$$

$$
\begin{aligned}
f_{0}^{k+1} &= \frac{1}{4}\Big[2f_{-1}^k + 2f_{0}^k\Big]\\
f_{1}^{k+1} &= \frac{1}{4}\Big[\Big(\frac{2}{9}-\eta\Big)f_{-1}^k + \Big(\frac{32}{9}+\eta\Big)f_{0}^k + \Big(\frac{2}{9}-\eta\Big)f_1^k\Big].
\end{aligned}
$$

This implies

$$
\begin{pmatrix} f_{-2}^{k+1}\\ f_{-1}^{k+1}\\ f_{0}^{k+1}\\ f_{1}^{k+1}\end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0\\ \frac{1}{18}-\frac{1}{4}\eta & \frac{8}{9}+\frac{1}{2}\eta & \frac{1}{18}-\frac{1}{4}\eta & 0\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{1}{18}-\frac{1}{4}\eta & \frac{8}{9}+\frac{1}{2}\eta & \frac{1}{18}-\frac{1}{4}\eta \end{pmatrix} \begin{pmatrix} f_{-2}^{k}\\ f_{-1}^{k}\\ f_{0}^{k}\\ f_{1}^{k}\end{pmatrix}.
$$

This further implies

$$
f_i^{k+1} = Sf_i^k,
$$

where

$$
S = \begin{pmatrix} 2 & 2 & 0 & 0\\ \frac{2}{9}-\eta & \frac{32}{9}+\eta & \frac{2}{9}-\eta & 0\\ 0 & 2 & 2 & 0\\ 0 & \frac{2}{9}-\eta & \frac{32}{9}+\eta & \frac{2}{9}-\eta \end{pmatrix}.
$$

The eigenvalues of the matrix above are $1, \frac{7}{18}+\frac{1}{2}\eta, \frac{1}{18}-\frac{1}{4}\eta, \frac{1}{2}$.
The eigenvectors of matrix S corresponding to these eigenvalues are named as follows:

$$Q = \begin{pmatrix} 1 & -\frac{27(9\eta+4)}{81\eta^2-198\eta-284} & 0 & -\frac{1}{2} \\ 1 & -\frac{3(9\eta+2)}{81\eta^2-198\eta-284} & 0 & 0 \\ 1 & -\frac{27(9\eta+4)}{81\eta^2-198\eta-284} & 0 & -\frac{1}{2} \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

This implies

$$Q^{-1} = \begin{pmatrix} \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & -\frac{9}{-11+9\eta} & \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & 0 \\ \frac{1}{6}\frac{81\eta^2-198\eta-284}{81\eta^2-63\eta-44} & -\frac{1}{3}\frac{81\eta^2-198\eta-284}{81\eta^2-63\eta-44} & \frac{1}{6}\frac{81\eta^2-198\eta-284}{81\eta^2-63\eta-44} & 0 \\ \frac{1}{3}\frac{-2+9\eta}{-11+9\eta} & \frac{1}{3}\frac{16+9\eta}{4+9\eta} & -\frac{26+45\eta}{4+9\eta} & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}.$$

For the decomposition of matrix S, we need Δ where Δ is a scalar matrix in which eigenvalues are arranged diagonally. Therefore,

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{7}{18}+\frac{1}{2}\eta & 0 & 0 \\ 0 & 0 & \frac{1}{18}-\frac{1}{4}\eta & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

So

$$\Delta^k = \begin{pmatrix} 1^k & 0 & 0 & 0 \\ 0 & \left(\frac{7}{18}+\frac{1}{2}\eta\right)^k & 0 & 0 \\ 0 & 0 & \left(\frac{1}{18}-\frac{1}{4}\eta\right)^k & 0 \\ 0 & 0 & 0 & \left(\frac{1}{2}\right)^k \end{pmatrix}.$$

By applying $k \to \infty$, all diagonal entries approach zero except the first diagonal entry 1.000. Now the above diagonal matrix becomes

$$\lim_{k\to\infty} \Delta^k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $f^{k+1} = Sf^k ... f^{k+1} = S^k f^0andf^k = S^k f^0$. As $k \to \infty$

$$f^{\infty} = Q(\lim_{k\to\infty} \Delta^k)Q^{-1}f^0.$$

$$Q(\lim_{k\leftarrow\infty}\Delta^k)Q^{-1} = \begin{pmatrix} \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & -\frac{9}{-11+9\eta} & \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & 0 \\ \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & -\frac{9}{-11+9\eta} & \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & 0 \\ \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & -\frac{9}{-11+9\eta} & \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & 0 \\ \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & -\frac{9}{-11+9\eta} & \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & 0 \end{pmatrix}.$$

$$\begin{pmatrix} f^{\infty}_{-2} \\ f^{\infty}_{-1} \\ f^{\infty}_{0} \\ f^{\infty}_{1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & -\frac{9}{-11+9\eta} & \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & 0 \\ \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & -\frac{9}{-11+9\eta} & \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & 0 \\ \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & -\frac{9}{-11+9\eta} & \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & 0 \\ \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & -\frac{9}{-11+9\eta} & \frac{1}{2}\frac{-2+9\eta}{-11+9\eta} & 0 \end{pmatrix} \begin{pmatrix} f^{0}_{-2} \\ f^{0}_{-1} \\ f^{0}_{0} \\ f^{0}_{1} \end{pmatrix}.$$

Thus limit stencil is:

$$\left[\frac{1}{2}\frac{-2+9\eta}{-11+9\eta}, -\frac{9}{-11+9\eta}, \frac{1}{2}\frac{-2+9\eta}{-11+9\eta}, 0\right]. \tag{18}$$

3.5 Summary of the characteristics of the schemes

The characteristics of the stationary schemes produced at $x = 1, 2, 3, 4$ are presented in Tables 1, 2, 3, 4. Properties of the other schemes can be proved analogously. From these tables, it is concluded that as the value of x increases, the characteristics of the schemes improve. Specifically, the scheme produced at $x = 4$ exhibits better characteristics than those produced at $x = 3$. In general, we observe that a larger value of x corresponds to a better scheme.

Sr. no	Characteristics	Conclusion
1	Convergence	convergent
2	Polynomial generation	1
3	Polynomial reproduction	1
4	Continuity	C^1 continuous in the interval $\eta \in (-\frac{7}{9}, \frac{2}{9})$
5	Limit stencil	$\left[\frac{1}{2}\frac{-2+9\eta}{-11+9\eta}, -\frac{9}{-11+9\eta}, \frac{1}{2}\frac{-2+9\eta}{-11+9\eta}\right]$

Table 1: A 3-point binary stationary scheme, where one rule involves 2 points and the other rule involves 3 points to find new points, was produced at $x = 1$.

Sr. no	Characteristics	Conclusion
1	Convergence	convergent
2	Polynomial generation	2
3	Polynomial reproduction	2
4	Continuity	C^2 continuous in the interval $\eta \in \left(-\frac{7}{9}, \frac{2}{9}\right)$
5	Limit stencil	$\left[\frac{1}{54} - \frac{1}{12}\eta, \frac{13}{27} + \frac{1}{12}\eta, \frac{13}{27} + \frac{1}{12}\eta, \frac{1}{54} - \frac{1}{12}\eta\right]$

Table 2: A 3-point binary stationary scheme, where both rules involve 3 points to find new points, was produced at $x = 2$.

Sr. no	Characteristics	Conclusion
1	Convergence	convergent
2	Polynomial generation	3
3	Polynomial reproduction	3
4	Continuity	C^3 continuous in the interval $\eta \in \left(-\frac{7}{9}, \frac{2}{9}\right)$
5	Limit stencil	$\left[-\frac{1}{108}\frac{81\eta^2-117\eta+22}{-65+9\eta}, \frac{1}{54}\frac{81\eta^2+450\eta-671}{-65+9\eta}, \frac{1}{54}\frac{81\eta^2+531\eta+2146}{-65+9\eta}, \frac{1}{54}\frac{81\eta^2+450\eta-671}{-65+9\eta}, -\frac{1}{108}\frac{81\eta^2-117\eta+22}{-65+9\eta}\right]$

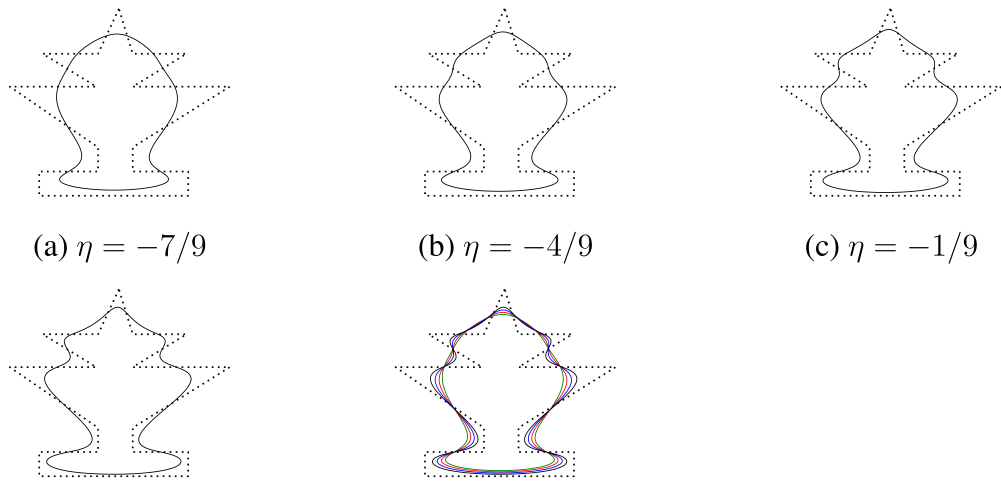
Table 3: A 4-point binary stationary scheme, where one rule involves 3 points and the other rule involves 4 points to find new points, was produced at $x = 3$.

Sr. no	Characteristics	Conclusion
1	Convergence	convergent
2	Polynomial generation	4
3	Polynomial reproduction	4
4	Continuity	C^4 continuous in the interval $\eta \in \left(-\frac{7}{9}, \frac{2}{9}\right)$
5	Limit stencil	$\left[\frac{1}{360}\eta^2 - \frac{17}{6480}\eta + \frac{13}{29160}, -\frac{1}{120}\eta^2 - \frac{163}{2160}\eta + \frac{143}{2430}, \frac{1}{180}\eta^2 + \frac{253}{3240}\eta + \frac{12851}{29160}, \frac{1}{180}\eta^2 + \frac{253}{3240}\eta + \frac{12851}{29160}, -\frac{1}{120}\eta^2 - \frac{163}{2160}\eta + \frac{143}{2430}, \frac{1}{360}\eta^2 - \frac{17}{6480}\eta + \frac{13}{29160}\right]$

Table 4: A 4-point binary stationary scheme, where both rules involve 4 points to find new points, was produced at $x = 4$.

4 Visual performance

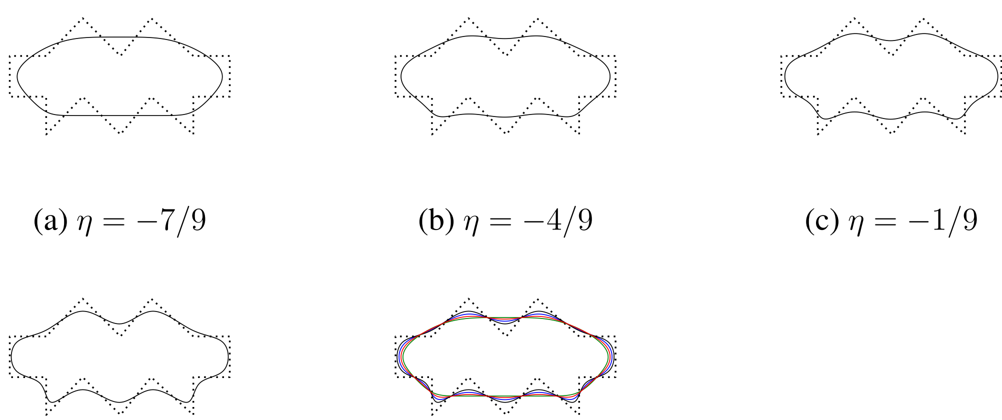
This section presents the visual performance of various schemes for different values of $\eta = -\frac{7}{9}, -\frac{4}{9}, -\frac{1}{9}$, and $\frac{2}{9}$. These parametric values are randomly chosen from the interval of continuity of the schemes. We explore different initial closed polygons and demonstrate the smooth curves obtained after applying the proposed subdivision schemes. Each scheme is applied to various initial sketches, resulting in a refined limit curve. The influence of the parameter is also illustrated in these figures. The smooth curves, referred to as shapes produced by the schemes, are depicted in Figures 1, 2, and 3.



(a) $\eta = -7/9$ (b) $\eta = -4/9$ (c) $\eta = -1/9$

(d) $\eta = 2/9$ (e) $\eta = -7/9,\ -4/9,\ -1/9$ and $2/9$

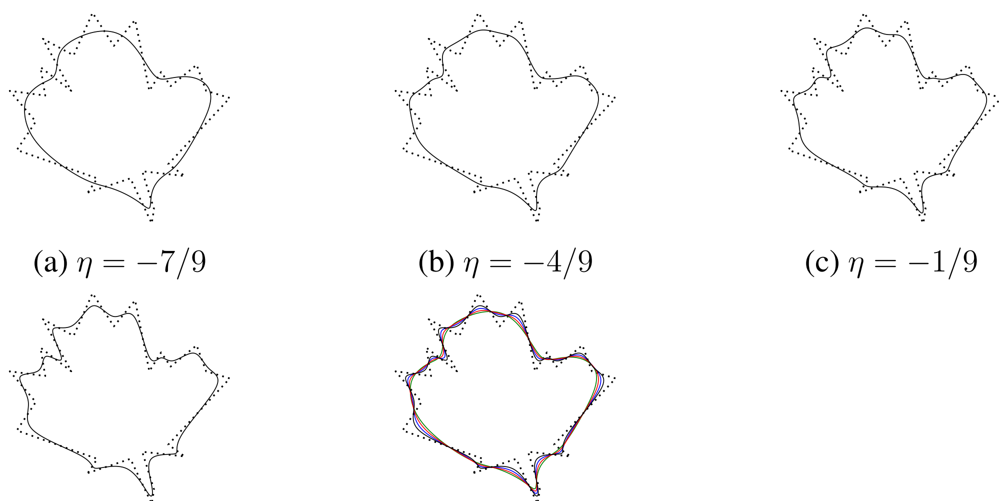
Figure 1: Present the visual performance of the 3-point scheme produced at $x = 2$ for different values of the parameter η. The dotted lines denote the initial polyline, while the black, blue, green, and red lines represent the curves produced with $\eta = -\frac{7}{9}, -\frac{4}{9}, -\frac{1}{9}$, and $\frac{2}{9}$ respectively.



(a) $\eta = -7/9$ (b) $\eta = -4/9$ (c) $\eta = -1/9$

(d) $\eta = 2/9$ (e) $\eta = -7/9,\ -4/9,\ -1/9$ and $2/9$

Figure 2: Present the visual performance of the 4-point scheme produced at $x = 3$ for different values of the parameter η. The dotted lines denote the initial polyline, while the black, blue, green, and red lines represent the curves produced with $\eta = -\frac{7}{9}, -\frac{4}{9}, -\frac{1}{9}$, and $\frac{2}{9}$ respectively.



(a) $\eta = -7/9$ (b) $\eta = -4/9$ (c) $\eta = -1/9$

(d) $\eta = 2/9$ (e) $\eta = -7/9,\ -4/9,\ -1/9$ and $2/9$

Figure 3: Present the visual performance of the 4-point scheme produced at $x = 4$ for different values of the parameter η. The dotted lines denote the initial polyline, while the black, blue, green, and red lines represent the curves produced with $\eta = -\frac{7}{9}, -\frac{4}{9}, -\frac{1}{9}$, and $\frac{2}{9}$ respectively.

5 Conclusion and future work

In this study, we introduce a diverse family of non-stationary subdivision schemes characterized by two parameters. One parameter determines the specific member of this scheme family, while the other parameter provides flexibility in shaping the curves produced. Additionally, leveraging the concept of asymptotic equivalence, we derived a separate family of stationary schemes. We proceeded to analyze these schemes comprehensively, examining their convergence behavior, polynomial generation degree, reproduction capabilities, continuity, and the structure of limit stencils. Practical applications of these schemes in generating smooth curves were explored in depth.

Our investigation concludes that as the parameter x increases, the characteristics of the schemes generally improve, suggesting that higher values of x correspond to better schemes. We also present the visual performance of various schemes for different values of the parameter η. By applying these schemes to different initial closed polygons, we demonstrate the production of smooth curves through iterative refinement. The influence of the parameter η on these curves is prominently illustrated in our figures.

Future research directions could explore extending these schemes into higher dimensions or investigating their applications in specialized fields such as computer graphics, where the generation of smooth curves remains a fundamental challenge.

Ethical Statement

We confirm that our manuscript adheres to the highest ethical standards. The research is original, properly cited, and not under consideration elsewhere. We declare no involvement of human subjects or animals in our study. We disclose any conflicts of interest. All listed authors contributed significantly and consented to submission.

Data Availability Statement

The authors confirm that the data supporting the findings of this study are available within the article.

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