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Some observability conditions for a fractional order descriptor system; a new approach

Fatima Khan

University of Education, Lahore, Multan Campus, Multan, Pakistan

Corresponding Email: fatima.khan@ue.edu.pk

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Abstract In this article, a descriptor systems involving singular and non-singular matrix $E_{n\times n}$ in the both input and output equation for the observability condition has been studied. The discussion is for both classical order derivative and Caputo's fractional order derivative.Firstly, the solution is obtained and further the observability conditions in form of Grammian matrices and rank conditions as per Kalman criteria are verified. These results are more generalized and can be verified for various special cases in the observability results for various types of descriptor systems.

Introduction

Descriptor systems play a vital role in all real life problems like agriculture[13], engineering[9], COVID-19 analysis[12], economics[17] etc. They frequently appear in solving methods of computational problems in analysis and design of linear systems. With the passage of time advancements are observed in various components of descriptors sytems like state, inputs, delays[10, 8] and outputs. These all components are associated together with a derivative. The nature of derivative may vary that is classical order derivative or fractional order derivative. Talking about the classical approach, some real life problems are not exactly interpreted as they can be interpreted in fractional calculus approach like [4, 14, 15, 16]. The most renowned fractional order derivatives are Riemann-Liouville, Caputo fractional order derivative, Caputo fabrizio [3] etc. Fractional calculus has given boost to its utilisation in real life problems and there are several applications of fractional calculus in fractional filters, fractional phase-locked loops, fractional wavelet, fractional predator–prey system, fractional electrical circuits etc. In this manner, one can see various types of articles by Younas et.al[20] in which they have discussed about the descriptor systems both the fractional and classical order derivatives. Talking about control theory, descriptor systems have nailed their role in this domain as well. In control systems the behaviour of inputs, outputs, state, feedback and delay are monitored to develop an efficient criterion or model for the accuracy of a dynamical system. There are several properties of control theory which can be studied in dynamical systems but the most highlighted are controllabilty and observability of the systems. Controllability is associated with the state behaviour of the system that how systems changes its state according to the input. On the other side observability is associated with the output behaviour of the system associated with the input and the unique state according to that output. To study all these, we need various types of descriptor systems and for controllability and observability, various types of mathematical techniques are adopted like diagonalization, eigen value analysis and the most popular among these all is Kalman criterion. Literature survey in control theory clearly indicates that singular fractional linear systems are not yet addressed in detail, as compared to non-singular fractional order linear systems. Moreover the emphasis is still more on controllability behaviours of dynamical systems as compared to the observability of the systems.Talking about optimum of our literature review, no related reports on the behavior of observability of the fractional dynamical system, especially in relation to their outputs as discussed in the literature. All of these factors have led us to the conclusion that sometimes the state equations has a multiple matrix namely E. Extending this problem, in this paper, we have studied descriptor system of such type that the matrix E appears in the output equation as well as the state space equation which is

$$Ey(t) = Cx(t) + Du(t) \tag{1}$$

We have studied such phenomenon under two cases:
Case 1 is about the invertible matrix E and the descriptor system involves classical order derivative.
Case 2 comprises of non-invertible matrix E and the descriptor system along with fractional order derivative. For non-invertible case, we have choosen the Drazin inverse approach[5]. This article consists of the following sections: In Section 3 we will recall basic definitions and some formulas. In Section 4, we have derived the necessary and sufficient conditions of observability for descriptor systems in both cases.

Basic Definitions

Here we first discuss some basics. Let us' take the descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), x(t_0) = x_0, \end{aligned} \tag{2}$$

where $x(t) \in \mathbf{R}^n$, is called the state vector, $\dot{x}(t) = (\frac{dx}{dt})$, $u(t) \in \mathbf{R}^m$ is the input vector, $y(t) \in \mathbf{R}^p$ is called output vector. The matrices E A, B, C and D are real constant matrices with relevant dimensions.

Definition The Reiman Loivelle fractional order derivative is defined as

$$D^{\alpha}_{a^+}[f(x)] = (1/(\Gamma(n-\alpha)))(\frac{d^n}{dx^n})\int_a^x (x-t)^{n-\alpha-1} f(t)dt, \tag{3}$$

$x \geq a$. It is note able that the derivative of constant is not zero in Riemann-Liouville fractional derivative .

Definition The matrix pencil of (E, A) of equation (2) is regular, $det(Ec - A) \neq 0$,for $c \in \mathbf{C}$.

Definition [6]If A is a square matrix of complex numbers, then index' of a matrix A, identified by $Ind(A)$,is the smallest positive integer q, such that $rankA^q = rankA^{q+1}$.

Definition [11]A Matrix$E^D \in \mathbf{R}^{n\times n}$, said to be the inverse Drazin of a matrix $E \in \mathbf{R}^{n\times n}$, if it fulfil' conditions given below

$$EE^D = E^D E, E^D E E^D = E^D and E^D E^{q+1} = E^q, \tag{4}$$

where q, is the' index of a matrix.

A square matrix's Drazin inverse exists,also unique. The Drazin inverse was launched in [7].

Algorithm [11]The procedure to calculate $E^D \in \mathbf{R}^{n\times n}$ for a matrix $E \in \mathbf{R}^{n\times n}$, are given below:

1.Discover the two matrices $V \in \mathbf{R}^{n\times r}, W \in \mathbf{R}^{r\times n}$, implying $rankV = r = rankE = rankW$ and

$$E = VW.$$

2.Determine the nonsingular matrix

$$WEV \in \mathbf{R}^{r\times r}.$$

3.Given by is inverse matrix for Drazin that is desired'.

$$E^D = V(WEV)^{-1}W.$$

If $Ind(A) = 1$, the inverse Drazin of A^D is called inverse of group and shown as A^* (see, e.g., [1]). Generaly, inverse Drazin can be written explicitly' in terms of the Jordan canonical form of A

$$A = S\begin{pmatrix} J & 0 \\ 0 & N \end{pmatrix} S^{-1}, A^D = S\begin{pmatrix} J^{-1} & 0 \\ 0 & 0 \end{pmatrix} S^{-1}$$

where J, includes the Jordan block for non zero eigen values, also N is nilpotent, with $N^k = 0$ and' $N^{k-1} \neq 0$. We can observe with the illustration of A^D[19]

$$R(A^D) = R(A^q), N(A^D) = N(A^q)$$

and

$$R^n = R(A^D) \oplus N(A^D). \tag{5}$$

There exists, $c \in \mathbf{C}$, that implies $det(Ec - A) \neq 0$, for $detE = 0$, with condition that the matrice's pencil (E, A) is regular.

In this article, firstly we have considered the descriptor system as

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ Ey(t) &= Cx(t) + Du(t), \end{aligned} \tag{6}$$

with $x(0) = x_O$. We have explored the descriptor system given below for generaliztion

$$ED_t^\alpha x(t) = Ax(t) + Bu(t)$$

$$Ey(t) = Cx(t) + Du(t)$$

with $x(0) = x_0$, where $0 < \alpha \leq 1$ and D_t^α is the Caputo's fractional-order derivative that is

$$D_t^\alpha f(t) = (1/(\Gamma(1-\alpha))) \int_0^t (t-\tau)^{-\alpha} (d/(ds)) f(s) ds.$$

Case 1: Matrix E is invertible in system (6):

Construction of solution for system (6):

Since the matrix E is invertible so, the first equation answered

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

in system (6) is as follows

$$x(t) = e^{E^{-1}At} x_0 + \int_{t_0}^{t_f} e^{E^{-1}A(t-\tau)} E^{-1} Bu(\tau) d\tau \tag{7}$$

AsE^{-1}exists so, the second equation of system (6) becomes

$$y = CE^{-1}x(t) + E^{-1}Du(t). \tag{8}$$

Substitute equation (7) in equation (8) we can write

$$y = CE^{-1}(e^{E^{-1}At} x_o + \int_{t_o}^{t_f} e^{E^{-1}A(t-\tau)} E^{-1} Bu(\tau) d\tau) + E^{-1}Du(t)$$

or we have

$$y(t) = CE^{-1} e^{E^{-1}At} x_o + CE^{-1} \int_{t_o}^{t_f} e^{E^{-1}A(t-\tau)} E^{-1} Bu(\tau) d\tau + E^{-1}Du(t) \tag{9}$$

Observability: System(6) is said observable on $[t_o, t_f]$, if initial state' x_o at $t = 0$ is uniquely illustrated by the associated input behavior $u(t)$ and output $y(t)$, $t \in [t_o, t_f]$.

Theorem: System (6) is called observable if and only if 'the Gramian matrix defined as

$$W_o[t_o,t_f]_{n\times n}=\int_{t_o}^{t=t_f}(e^{E^{-1}At})^T(CE^{-1})^T(CE^{-1})(e^{E^{-1}At})dt. \tag{10}$$

is invertible for $t_f>0$.

Proof Using equation (7) and (9) we define

$$\overline{y}(t)=y(t)-CE^{-1}\int_{t_o}^{t_f}e^{E^{-1}A(t-\tau)}E^{-1}Bu(\tau)d\tau-E^{-1}Du(t) \tag{11}$$

and simply we have

$$y(t)=CE^{-1}e^{E^{-1}At}x_o$$

.

It is obvious that we can equally estimate x_o from $y(t)$ as the observability of system (6). Due to the arbitrary nature of $\overline{y}(t)$ and x_o , an estimate of x_o from $y(t)$ is as below

$$y(t)=CE^{-1}e^{E^{-1}At}x_o$$

as $u(t)=0$.

If $W_o[t_o,t_f]$ is invertible, then $W_o^{-1}[t_o,t_f]$ exists and will be welldefined. Consequently, we may write the following for any arbitrarily chosen $y(t)$, such that $t_f>0$.

$$W_o^{-1}[t_o,t_f]\int_{t_o}^{t_f}(e^{E^{-1}A(t)})^T(CE^{-1})^Ty(t)dt$$

$$=W_o^{-1}[t_o,t_f]\int_{t_o}^{t_f}(e^{E^{-1}A(t)})^T(CE^{-1})^TCE^{-1}e^{E^{-1}At}x_o dt$$

$$=W_o^{-1}[t_o,t_f]W_o[0,t_f]x_o.$$

So, naturally,

$$W_o^{-1}[t_o,t_f]\int_{t_o}^{t_f}(e^{E^{-1}A(t)})^T(CE^{-1})^Ty(t)dt=x_o. \tag{12}$$

The left side of (12) is dependent on $y(t)$ such that $t\in[t_o,t_f]$, and (12)is an equation in linear algebra for x_o . Since $W_o[t_o,t_f]$ exists, initial state x_o can only be identified by the output of the system $y(t)$ that corresponds to it.

Now for the converse part, if $W_o^{-1}[t_o,t_f]$ does not exists for any $t_f>0$, then there exists $x_y\neq 0$ that is

$$x_y^TW_o[t_o,t_f]x_y=0$$

.

Consider $x_y = x_o$, for

$$\int_{t_o}^{t_f} y^T(t)y(t)dt$$

$$= x_o^T \int_{t_o}^{t_f} e^{E^{-1}A(t)^T}(CE^{-1})^T(CE^{-1})e^{E^{-1}A(t)}x_o dt = 0.$$

In light of this,

$$\int_{t_o}^{t_f} ||y(t)||^2 dt = 0.$$

Conclusively

$$y(t) = (CE^{-1})e^{E^{-1}A(t)}x_o = 0.$$

As we have assumed an observable system and it gives that $x_o = 0$. Which becomes a contradiction and hence proved that $W_o[t_o, t_f]$ is invertible.

Theorem: System is said' observable on $[t_o, t_f]$ if and only if

$$rankQ_E = \begin{pmatrix} CE^{-1} \\ (CE^{-1})(E^{-1}A) \\ . \\ . \\ . \\ (CE^{-1})(E^{-1}A)^{n-1} \end{pmatrix} = n.$$

Proof: From above Theorem, we have

$$y(t) = CE^{-1}e^{E^{-1}A(t)}x_o.$$

In this expression x_o is determined uniquely via $y(t)$ iff the term $CE^{-1}e^{E^{-1}A(t)}$ is invertible. Cayley's Hamilton theorem allows us to write

$$CE^{-1}e^{E^{-1}A(t)} = CE^{-1}\sum_{i=0}^{n-1} l_i(t)(E^{-1}A)^i.$$

Which yields at $t = t_f$

$$CE^{-1}e^{E^{-1}A(t)} = \begin{pmatrix} l_o(t_f) & l_1(t_f) & . & . & . & l_{n-1}(t_f) \end{pmatrix} \begin{pmatrix} CE^{-1} \\ (CE^{-1})(E^{-1}A) \\ . \\ . \\ . \\ (CE^{-1})(E^{-1}A)^{n-1} \end{pmatrix}$$

From this equation the only condition which holds is that$CE^{-1}e^{E^{-1}A(t)}$ is invertible if and only if

$$rank \begin{pmatrix} CE^{-1} \\ (CE^{-1})(E^{-1}A) \\ . \\ . \\ . \\ (CE^{-1})(E^{-1}A)^{n-1} \end{pmatrix} = n.$$

Thus the system is observable on $[t_o, t_f]$ if and only if $rank(Q_w) = n$.

Case 2: $detE = 0$. Now consider that the descriptor system involves the fraction order Caputo's fractional order derivative and E^{-1} does not exists. So, the descriptor system is of the following form

$$\begin{aligned} ED_t^{\alpha}x(t) = Ax(t) + Bu(t) \\ Ey(t) = Cx(t) + Du(t), x(t_o) = x_o. \end{aligned} \tag{13}$$

Multiplying the system (13) with the regular pencil, we obtain the following equivalent system

$$\overline{E}D_t^{\alpha}x(t) = Ax(t) + Bu(t),$$

$$Ey(t) = C(t) + Du(t)$$

where

$$\begin{aligned} \overline{E} = [Ec - A]^{-1}E, \overline{A} = [Ec - A]^{-1}A, \overline{B} = [Ec - A]^{-1}B. \\ \overline{C} = C[Ec - A]^{-1} and \overline{D} = D[Ec - A]^{-1}. \end{aligned} \tag{14}$$

As E is not invertible, the answer to the first equation of system (13) is as follows

$$x(t) = \Phi_{\alpha,1}(\overline{E}^D\overline{A}, t)w + \int_{t_o}^{t} \Phi_{\alpha,\alpha}(\overline{E}^D\overline{A}, t-\tau)\overline{E}^D\overline{B}u(\tau)d\tau + (\overline{E}\overline{E}^D - \mathbf{I})\overline{A}^D\overline{B}u(t), \tag{15}$$

where $\Phi_{\alpha,\beta}(E^{-1}A, t) = \sum_{k=0}^{\infty} \frac{(E^{-1}A)^k t^{\alpha k+\beta-1}}{(\Gamma(\alpha k+\beta))}$ is called the state transfer matrix and $\Gamma(.)$ is a

Gamma-function.

Definition: [15] For a random square matrix A, the Mittag-Leffler two-parameters function is

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta > 0 \tag{16}$$

in particular, $E_{\alpha,1}(A) = E_{\alpha}(A)$, with $\beta = 1$. The following describes how state transfer matrices relate to Mittag-Leffler.

$$\Phi_{\alpha,\beta}(A,t) = t^{\beta-1} E_{\alpha,\beta}(At^{\alpha})$$

Also, verifying that is simple.[18]

$$D_t^{\alpha} \Phi_{\alpha,\beta}(A,t) = A\Phi_{\alpha,\beta}(A,t). \tag{17}$$

The second equation

$$Ey(t) = Cx(t) + Du(t),$$

from the system (13) can be expressed as the following expression [2]

$$y = \overline{C}\overline{E}^D x(t) - (\overline{E}\overline{E}^D - \mathbf{I})w \tag{18}$$

where, $w \in \mathbf{R}(A^{k-1}) + N(A)$. Substitute equation (17) in (18) we can write

$$y = \overline{C}\overline{E}^D(\Phi_{\alpha,1}(\overline{E}^D\overline{A},t)w + \int_{t_o}^{t} \Phi_{\alpha,\alpha}(\overline{E}^D\overline{A},t-\tau)\overline{E}^D\overline{B}u(\tau)d\tau$$

$$+(\overline{E}\overline{E}^D - \mathbf{I})\overline{A}^D\overline{B}u(t) - (\overline{E}\overline{E}^D - \mathbf{I})w)$$

or

$$y(t) = \overline{C}\overline{E}^D\Phi_{\alpha,1}(\overline{E}^D\overline{A},t)w + \overline{C}\overline{E}^D \int_{t_o}^{t} \Phi_{\alpha,\alpha}(\overline{E}^D\overline{A},t-\tau)\overline{E}^D\overline{B}u(\tau)d\tau$$

$$+\overline{C}\overline{E}^D(\overline{E}\overline{E}^D - \mathbf{I})\overline{A}^D\overline{B}u(t) - (\overline{E}\overline{E}^D - \mathbf{I})w)$$

or

$$\begin{array}{c} y(t) = \overline{C}\overline{E}^D\Phi_{\alpha,1}(\overline{E}^D\overline{A},t)w + \overline{C}\overline{E}^D \int_{t_o}^{t} \Phi_{\alpha,\alpha}(\overline{E}^D\overline{A},t-\tau)\overline{E}^D\overline{B}u(\tau)d\tau \\ -(\overline{E}\overline{E}^D - \mathbf{I})w) \end{array} \tag{19}$$

Theorem: System (13) is called observable if and only if the Grammian matrix

$$W_D[0,t_f] = \int_{t_o}^{t_f} (\Phi_{\alpha,1}(\overline{E}^D\overline{A},t))^T(\overline{C}\overline{E}^D)^T(\overline{C}\overline{E}^D\Phi_{\alpha,1}(\overline{E}^D\overline{A},t))dt \tag{20}$$

is invertible.

Proof: As we have obtained the result as

$$y(t) = \overline{CE}^D \Phi_{\alpha,1}(\overline{E}^D\overline{A}, t)w + \overline{CE}^D \int_{t_o}^{t} \Phi_{\alpha,\alpha}(\overline{E}^D\overline{A}, t-\tau)\overline{E}^D\overline{B}u(\tau)d\tau$$

$$-(\overline{EE}^D - \mathbf{I})w)$$

Using the property$\overline{E}^D(\overline{EE}^D - \mathbf{I}) = 0$. Now we take $\overline{y}(t)$ as

$$\overline{y}(t) = y(t) - \overline{CE}^D \int_{t_o}^{t} \Phi_{\alpha,\alpha}(\overline{E}^D\overline{A}, t-\tau)\overline{E}^D\overline{B}u(\tau)d\tau$$

$$+(\overline{EE}^D - \mathbf{I})w)$$

so equation (19) becomes

$$\overline{y}(t) = \overline{CE}^D \Phi_{\alpha,1}(\overline{E}^D\overline{A}, t)w \tag{21}$$

This is equivalent to an estimate of initial state by the corresponding output.

Now if the gramian matrix is invertible than we have

$$W_o^{-1}[t_o, t_f] \int_{t_o}^{t_f} (\Phi_{\alpha,1}(\overline{E}^D\overline{A}, t))^T (\overline{CE}^D)^T y(t)dt,$$

$$= W_o^{-1}[t_o, t_f] \int_{t_{o]}}^{t_f} (\Phi_{\alpha,1}(\overline{E}^D\overline{A}, t))^T (\overline{CE}^D)^T (\overline{CE}^D \Phi_{\alpha,1}(\overline{E}^D\overline{A}, t))w dt$$

$$= W_o^{-1}[t_o, t_f]W_o[0, t_f]w = w.$$

So we have

$$W_o^{-1}[t_o, t_f] \int_{t_o}^{t_f} (\Phi_{\alpha,1}(\overline{E}^D\overline{A}, t))^T (\overline{CE}^D)^T y(t)dt = w_o \tag{22}$$

The L.H.S of equation (22) depends' upon y and it is linear-algebraic equation, in terms of w_o which can be uniquely determined. Now the other side $z^T W_o[t_o, t_f]z = 0$. If we choose $z = w$ then

$$\int_{t_o}^{t_f} (y(t))^T (y(t))dt = w_o W_o[0, t_f]w_o = 0$$

which results into

$$\int_{t_o}^{t_f} ||y(t)||^2 dt = 0$$

which follows that $w_o = 0$ that contradicts our result.

Theorem: The system (13) is observable, if and only if the following rank condition satisfies that is

$$rank \begin{pmatrix} \overline{CE}^D \\ (\overline{CE}^D)(\overline{E}^D\overline{A}) \\ . \\ . \\ . \\ (\overline{CE}^D)(\overline{E}^D\overline{A})^{n-1} \end{pmatrix} = n.$$

Proof: This proof is similar to proof 2.

Example: Consider the following system,

$$\begin{aligned} ED_t^{\alpha}x(t) = Ax(t) + Bu(t) \\ Ey(t) = Cx(t), \end{aligned} \tag{23}$$

with $x(0) = 0$.

Here we have

$$E = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & -1 \end{bmatrix}$$

Firstly, we will evaluate the matrix pencil (E,A),which is 'regular', for

$$c = 1.$$

So,

$$(E - A)^{-1} = \begin{bmatrix} (4/(29)) & -(1/(29)) \\ -(1/(29)) & -(7/(29)) \end{bmatrix}$$

Now calculating the following matrices

$$E = E(Ec - A)^{-1};$$

$$\overline{E} = \begin{bmatrix} (5/(29)) & (6/(29)) \\ -(5/(29)) & -(6/(29)) \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} -((24)/(29)) & (6/(29)) \\ -(5/(29)) & -((35)/(29)) \end{bmatrix}$$

$$\overline{C} = \begin{bmatrix} -(6/(29)) & -((42)/(29)) \end{bmatrix}$$

and the Drazin inverse for matrix E as per above mentioned method is

$$\overline{E}^D = \begin{bmatrix} (5/(24389)) & (6/(24389)) \\ -(5/(24389)) & -(6/(24389)) \end{bmatrix}$$

Also

$$\overline{CE}^D = \begin{bmatrix} ((25)/(707281)) & ((30)/(707281)) \end{bmatrix}$$

And

$$(\overline{CE}^D)(\overline{E}^D\overline{A}) = \begin{bmatrix} ((750)/(500246412961)) & ((900)/(500246412961)) \end{bmatrix}$$

Now the rank criterion for a 2×2 system becomes

$$rank \begin{bmatrix} \overline{CE}^D \\ (\overline{CE}^D)(\overline{E}^D\overline{A}) \end{bmatrix}$$

Substituting all these calculated values, we have

$$rank = \begin{bmatrix} ((25)/(707281)) & ((30)/(707281)) \\ ((750)/(500246412961)) & ((920)/(500246412961)) \end{bmatrix}$$

So the system (23) is observable.

Conflict of interest: "There is no conflict of interest".

References

[1] Abraham Berman and Robert J Plemmons. *Nonnegative matrices in the mathematical sciences*. SIAM, 1994.

[2] Haifa Bin Jebreen, Yurilev Chalco-Cano, et al. An improved computationally efficient method for finding the drazin inverse. *Discrete Dynamics in Nature and Society*, 2018, 2018.

[3] Li C, Qian D, and Chen Y. On riemann-liouville and caputo derivative. *Discrete Dynamics in Nature and Society*, 2011.

[4] K Nishimoto Fractional Calculus. Koriyama, 1984.

[5] Stephen L Campbell and Carl D Meyer. *Generalized inverses of linear transformations*. SIAM, 2009.

[6] L Dai. Singular linear systems. *Lecture Notes in Control and Information Sciences, Berlin, German: Spring-Verlag*, 1989.

[7] MP Drazin. Pseudo-inverses in associative rings and semigroups. *The American mathematical monthly*, 65(7):506–514, 1958.

[8] MS Fofana, BQ Cabell, N Vawter, MF Usilton, and PB Ryba. Illustrative applications of the theory of delay dynamical systems. *Mathematical and computer modelling*, 37(12-13):1371–1382, 2003.

[9] Tadeusz Kaczorek. Controllability and observability of linear electrical circuits. *Przegląd Elektrotechniczny*, 87(9a):248–254, 2011.

[10] Tadeusz Kaczorek et al. Realization problem for positive linear systems with time delay. *Mathematical Problems in Engineering*, 2005(4):455–463, 2005.

[11] Tadeusz Kaczorek and Krzysztof Rogowski. *Fractional linear systems and electrical circuits*. Springer, 2014.

[12] Sarbaz HA Khoshnaw, Muhammad Shahzad, Mehboob Ali, and Faisal Sultan. A quantitative and qualitative analysis of the covid–19 pandemic model. *Chaos, Solitons & Fractals*, 138:109932, 2020.

[13] Sajib Mandal, Sebastian Oberst, Md Haider Ali Biswas, and Md Sirajul Islam. Dynamic analysis and control of a rice-pest system under transcritical bifurcations. *PeerJ*, 11:e16083, 2023.

[14] Keith Oldham and Jerome Spanier. *The fractional calculus theory and applications of differentiation and integration to arbitrary order*. Elsevier, 1974.

[15] I Podlubny. Fractional differential equations academic press, san diego, 1999. *Search in*, 1999.

[16] I Podlubny. Fractional calculus appl. *Anal*, 5:367, 2002.

[17] Jelena Stanojevic and Katarina Kukić. *Dynamical systems in economics*, volume 1926. 01 2018.

[18] Jiang Wei. The controllability of fractional control systems with control delay. *Computers & Mathematics with Applications*, 64(10):3153–3159, 2012.

[19] Wolfgang Wülling. The drazin inverse of a singular, unreduced tridiagonal matrix. *Linear Algebra and its Applications*, 439(10):2736–2745, 2013.

[20] A Younus, I Javaid, and A Zehra. On controllability and observability of fractional continuous-time linear systems with regular pencils. *Bulletin of the Polish Academy of Sciences. Technical Sciences*, 65(3):297–304, 2017.